**Relativistic Bosons**

**Special Relativity for single particle (review)**

So we’ll recall that our attempt at a relativistically invariant theory for bosons (or well, spinless particles) gave us the KG equation, which is as follows, using natural units ℏ = 1, c = 1:



(using the η = (+1, -1, -1, -1) metric) But of course there were issues with this theory. One was that we couldn’t associated with it a positive definite probability density. We didn’t explore this issue per se´, but it also suffers from the same negative energy solutions that the Dirac equation did. Another is that it is incapable of describing particle creation/annihilation. And so we look for a *field* theory of these particles. In this view, the particles will emerge as excitations of the field, in the same way that phonons emerge as excitations of an elastic field, and photons as excitations of the EM field.

**\*complex\* bosonic field**

We can generalize our previous example, and allow the bosonic field to take on complex values. This would make sense from 2nd quantization point of view as the wavefunction from which the field is heuristically constructed can take on complex values. But it would be less clear what that physically means per se´. Anyway, so define a complex field,



where subscripts refer to real/imaginary parts. And note that the real/imaginary parts of φ are Hermitian, and so,



Then we can write down the H,

 

and in terms of real/imaginary parts, this would look like,



So basically we’re just describing two real independent fields. And we can see that the canonical commutation relations:



are consistent with the ‘complex’ ones. L would be,

 

And broken down into real and imaginary parts, these would be as follows:



And again, the commutation relations:



are consistent with the complex ones. Once again the easiest way to get the excitations is via the equation of motion. So,



So our equation of motion for the field is



Doing the same for the complex field simply yields the complex conjugate of this equation.



To solve this we take the spatial Fourier transform of both sides. Keeping in mind that there are fields that could have boundary conditions, etc. imposed on them, it is useful to keep in mind that what we’re really doing is writing out our equation as



Then we look for the eigenfunctions of the spatial operator/BC.



And these ‘are’ ei**k**·**x**, with eigenvalues ωk = √(|**k**|2 + m2), just like before. So we then expand the solution in terms of the eigenfunctions…



Plugging this into the equation of motion,



we find the solution



So our solution so far is:



Now we’ll want the properties, commutation relations among the φ0(**k**)’s. We’ll start with:



which implies,



And the commutation relations can be inferred as usual. Evaluating at t = 0…



We see we need,



Now we’ll put the fields in the FFE form. Judging from the form of φ, we see that we can identify the excitation as ωk, and the annihilation operators as,



The use of -**k** in the latter is just a convention to make it look pretty in the end. Since a, b annihilate excitations and the excitations are the same whether we have **k** or -**k**, the definition is kind of arbitrary. The normalization is conventionally determined via:



The one for the b’s will work out the same. And we can verify that a’s and b’s commute. For instance,



So we have:



and so we can write our field(s) as:



and,



So,



If we plug these back into H, then we should get, sans an infinite constant,



**Conservation Laws**

We can do the same thing as for the real boson and derive an expression for the total energy and momentum. We should get, neglecting infinite constants (and note superscript *m* stands for field or complex conjugate of field):



and



Now let’s consider an interesting continuous field symmetry. We might note the field has U(1) symmetry, meaning the Lagrangian is invariant when we multiply the field by a pure phase: φ → φ(λ) = eiλφ. The associated conservation law is as in that file, but here Sμ = 0, because the Lagrangian doesn’t change at all under this transformation. So



where the α index denotes: α = 1 → field, α = 2 → field complex conjugate. Now expanding φ(λ) in a Taylor series, we have δφ = iλφ. Then using,



And forming ρ we have:



and forming **j** we have:



Sans the iλ, we will recognize these as the probability density and current in the KG equation. Before, that interpretation was problematic because ρ could be negative, which a true probability density cannot be. Here we see the resolution. ρ is actually charge density, j current density, where a’s are associated with positive charge, and b’s with negative charge. We can see this by filling the FFE into ρ and j. For Q we’ll get, dropping the iλ:



Ignoring the ab and a†b†, because they’ll vanish under any expectation, and also ignoring the infinite constant we’d get when we permute aa†, we have:



This is just adding up all of the a’s and subtracting all of the b’s. The clear interpretation is that a charge +1 is associated with the a’s, and a charge -1 with the b’s. And this charge is the conserved quantity. We can work out the (well, the total) current. This is:

